On Generalizations of Conics and on a Generalization of the Fermat-Torricelli Problem

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Conics are introduced usually in one of the two following ways:

- A conic $K_\rho^{(1)}$ is a set of points $(x, y) \in \mathbb{R}^2$ in the plane that satisfy a polynomial equation in variables x and y of maximum degree 2 [**8**, 385ff] ($a, b, c, d, e, \rho \in \mathbb{R}$):

$$K_\rho^{(1)} := \{(x, y) \in \mathbb{R}^2 | f(x, y) = \rho\}, \text{where } f(x, y) := ax^2 + bx + cxy + dy + ey^2.$$

- A conic $K_\rho^{(2)}$ is a set of points $\vec{x} := (x, y) \in \mathbb{R}^2$ in the plane that have a constant weighted sum of distances to two points $\vec{b}_1, \vec{b}_2 \in \mathbb{R}^2$ [**23**, 75ff] ($\alpha_i = \pm 1$, $\rho \in \mathbb{R}$):

$$K_\rho^{(2)} := \{(x, y) \in \mathbb{R}^2 | g(x, y) = \rho\}, \text{where } g(x, y) := \alpha_1|\vec{x} - \vec{b}_1| + \alpha_2|\vec{x} - \vec{b}_2|.$$

Depending on the choice of the coefficients a to e, respectively, or the coordinates $\vec{b}_1, \vec{b}_2$ and weights α_i one gets ellipses, parabolas, and hyperbolas and it is possible to convert the curve representations into each other. The points $\vec{b}_i$ are called *focal points* or *foci*, and we call these conics *classical conics* in contrast to the extended definitions that follow.

We generalize the second definition by allowing more than two focal points, weights other than $\alpha_i = \pm 1$ [**11**], [**15**], [**16**], and point sets in higher dimensions in arbitrary vector norms [**4**], [**5**]. Similar generalizations have been given various names by different authors: trillipse, egglipse, Eikurve/fläche, oval, ovaloid, polyconic, multifocal curve, polyzomal curve. In [**17**] we have shown for positive weights that the interior of these generalized conics is convex, that the sets are ordered by inclusion and that there always exists a smallest non-empty set with respect to this ordering, the set of so-called *Fermat-Torricelli points*. Furthermore we have shown that generalized conics in differentiable norms are differentiable if and only if they don't contain the focal points.

We give a classification of generalized conics analogous to the classical ones due to the weights and locations of the focal points. We also examine generalized conics with respect to several properties that hold for classical conics. For various mechanical and optical properties of classical conics, we show which of them apply to generalized conics. From this treatment we get another generalization of conics in the plane. The results in this article are also interesting in connection with the problem of determining the Fermat-Torricelli points [**1**], [**6**], [**12**]. We construct examples of configurations of focal points in certain norms where the Fermat-Torricelli set contains points that do not lie in the convex hull of the focal points [**7**].

1. DEFINITION OF GENERALIZED CONICS. Recall that a norm in $\mathbb{R}^d$ is a map $\|\cdot\|: \mathbb{R}^d \to \mathbb{R}_0^+$ that is *homogeneous* and obeys the *triangle inequality*:

$$\|t\vec{x}\| = |t|\,\|\vec{x}\|, \quad \|\vec{x}+\vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|, \quad \vec{x}, \vec{y} \in \mathbb{R}^d, \quad t \in \mathbb{R}.$$

Its *unit sphere* contains all $\vec{x} \in \mathbb{R}^d$ with $\|\vec{x}\| = 1$. Standard examples for $\|\cdot\|$ are the so-called p-norms $(1 \le p < \infty)$ and the ∞-norm. Denoting by x_k, b_{ik} the coordinates of points $\vec{x}, \vec{b}_i$ in $\mathbb{R}^d$, we have:

$$\|\vec{x} - \vec{b}_i\|_p := \sqrt[p]{\sum_{k=1}^{d} |x_k - b_{ik}|^p}\,, \quad \|\vec{x} - \vec{b}_i\|_\infty := \max_{k=1,\dots,d} \{|x_k - b_{ik}|\}.$$

Now we can define the *generalized radius* $\rho(\vec{x})$ of a point $\vec{x} \in \mathbb{R}^d$ as the sum of weighted distances to n focal points $\vec{b}_1, \dots, \vec{b}_n \in \mathbb{R}^d$ with weights $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ and point sets $K_{\rho,n}, E_{\rho,n}$:

$$\rho(\vec{x}) := \sum_{i=1}^{n} \alpha_i \|\vec{x} - \vec{b}_i\|, \qquad \begin{aligned} K_{\rho,n} &:= \{\vec{x} \in \mathbb{R}^d |\, \rho(\vec{x}) = \rho\}, \\ E_{\rho,n} &:= \{\vec{x} \in \mathbb{R}^d |\, \rho(\vec{x}) \le \rho\}. \end{aligned} \tag{1}$$

$K_{\rho,n}$ is called a *generalized conic*. Just as the classical conics are niveau lines of the function g, the generalized conics are niveau lines, respectively, niveau surfaces of the generalized radius.

This definition holds for arbitrary dimensions. For $n = 1$, $\alpha_1 > 1$, we obtain balls in d-space, and for $n = 2$, $\alpha_i > 0$, we obtain ellipses $(\alpha_1 = \alpha_2)$ and egg-shaped curves $(\alpha_1 \neq \alpha_2)$ **[24]**. For the figure captions we use a notation in which the weights of the focal points are an additional last coordinate $(\vec{b}_{i,1}, \dots, \vec{b}_{i,d}, \alpha_i) \in \mathbb{R}^{d+1}$.

2. COMPACTNESS AND CONVEXITY-TOPOLOGICAL PROPERTIES. In this section we show that the sets $E_{\rho,n}$ are convex if all weights are positive, i.e., they can be viewed as the interior of *generalized ellipses*. For the reader's convenience, we then recall further properties of generalized conics that were proved in **[17]**.

The *convex hull* of two points $\vec{x}, \vec{y} \in \mathbb{R}^d$ is the line segment that connects these points:

$$\operatorname{conv}\{\vec{x}, \vec{y}\} := \left\{\vec{t} \in \mathbb{R}^d | \vec{t} = t\vec{x} + (1-t)\vec{y}, t \in [0,1]\right\}.$$

A set of points M is called *convex*, if for all pairs $(\vec{x}, \vec{y}) \in M \times M$ also their convex hull is contained in M. For $\vec{b} \in \mathbb{R}^d$ and $\vec{t} \in \operatorname{conv}\{\vec{x}, \vec{y}\}$ we have

$$\begin{aligned} \|\vec{p} - \vec{t}\| &= \left\|\vec{p} - (t\vec{x} + (1-t)\vec{y})\right\| = \left\|t(\vec{p} - \vec{x}) + (1-t)(\vec{p} - \vec{y})\right\| \\ &\le t\|\vec{p} - \vec{x}\| + (1-t)\|\vec{p} - \vec{y}\|. \end{aligned}$$

The $E_{\rho,n}$ are convex sets (see Figure 1): if $\vec{x}, \vec{y} \in E_{\rho,n}$ and $\vec{t} \in \operatorname{conv}\{\vec{x}, \vec{y}\}$ then also $\vec{t} \in E_{\rho,n}$, since

$$\begin{aligned} \rho(\vec{t}) &= \sum_{i=1}^{n} \|\vec{p}_i - \vec{t}\| \le t \sum_{i=1}^{n} \|\vec{p}_i - \vec{x}\| + (1-t) \sum_{i=1}^{n} \|\vec{p}_i - \vec{y}\| \\ &\le t\rho(\vec{x}) + (1-t)\rho(\vec{y}). \end{aligned} \tag{2}$$

Let $B = \{\vec{b}_1, \dots, \vec{b}_n\} \subset \mathbb{R}^d$ be a configuration of n focal points with weights $\alpha_1, \dots, \alpha_n$. Let $\underline{\rho}$ denote the infimum of generalized radii for which the set $E_{\rho,n}$ is

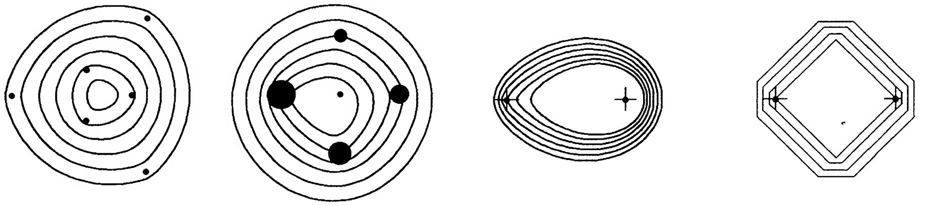

Figure 1. Examples of generalized ellipses: (left) convexity for non-convex configurations of focal points in the plane with $\alpha_i = 1$, respectively, $\alpha_i = i$ and (right) non-differentiable points with $\alpha_1 = 2$, $\alpha_2 = 3$, 2-norm, respectively, $\alpha_1 = \alpha_2 = 1$, ∞-norm.

non-empty. Then $E_{\underline{\rho}, n}$ is called a *Fermat-Torricelli set*, i.e.,

$$\underline{\rho} = \inf\{\rho | E_{\rho, n} \neq \varnothing\}, \quad E_{\underline{\rho}, n} = \left\{\vec{x} \in \mathbb{R}^d | \rho(\vec{x}) = \underline{\rho}\right\} = K_{\underline{\rho}, n}.$$

If all weights are positive, then we have [17]:

Theorem 2.1. *The sets $E_{\rho, n}$ are convex and compact.*

Theorem 2.2. *For finite sets of focal points in each norm the Fermat-Torricelli set is non-empty.*

Theorem 3.1. *For $\rho > \underline{\rho}$ and with $D_{\rho, n} := \{\vec{x} \in \mathbb{R} | \rho(\vec{x}) < \rho\}$, we have*

$$D_{\rho, n} = \left(E_{\rho, n}\right)^{\circ}, \quad K_{\rho, n} = \partial E_{\rho, n}.$$

Theorem 3.2. *The generalized conics $K_{\rho, n}$ are compact.*

Theorem 3.3. *Let $\|\cdot\|$ be a norm with a unit sphere that has in all tangential directions strictly positive normal curvature. Suppose $|E_{\underline{\rho}, n}| > 1$, i.e., at least two distinct Fermat-Torricelli points exist. Then all focal points are located on the straight line through these two Fermat-Torricelli points.*

Theorem 4.1. *Suppose the norm $\|\cdot\|$ is m-times differentiable on $\mathbb{R}^d \setminus \{\vec{0}\}$ and that $\rho > \underline{\rho}$. Then all $K_{\rho, n}$ that contain no focal points are compact $(d-1)$-dimensional C^m-manifolds. They are the boundary of convex compact sets, are homeomorphic to a $(d-1)$-sphere, and hence are connected for $d > 1$.*

It is important to note the restrictions in these theorems. For example, Figure 1 shows that generalized conics that contain focal points may have vertices there; such conics are not differentiable, even if $\|\cdot\|$ is. Also for $\rho = \underline{\rho}$, Figure 2 shows that $E_{\rho, n}$ may have a non-empty interior, so $(E_{\underline{\rho}, n})^{\circ} \neq D_{\underline{\rho}, n} = \varnothing$. Here $|E_{\underline{\rho}, n}| > 1$,

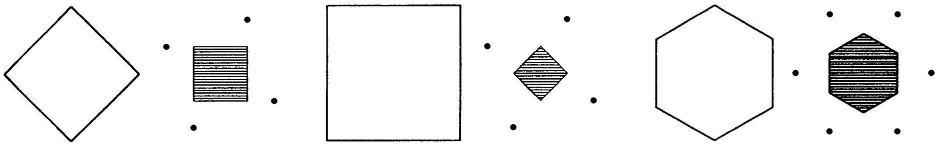

Figure 2. Examples of unit circles (on the left) and configurations of focal points (on the right) with Fermat-Torricelli sets with non-empty interior (dashed). From left to right: 1-norm, ∞-norm and 6-gon-norm [20].

although the focal points are not collinear. According to Theorem 4.1, generalized conics are $(d-1)$-dimensional manifolds. In the sequel the *1-dimensional case* means curves in $\mathbb{R}^2$; the *2-dimensional case* means surfaces in $\mathbb{R}^3$. Hence the Fermat-Torricelli set could contain more than one point if: i) the unit sphere of the norm contains line segments, see Figure 2, or ii) all focal points are contained in a 1-dimensional subspace. In the latter case, the Fermat-Torricelli set is the (weighted) *median* of all focal points, which may either be one of the focal points or a line segment that connects two of them. In the former case, the Fermat-Torricelli set may even contain points that do not lie in the convex hull of the focal points, as in the example on the very right in Figure 1, where the Fermat-Torricelli set consists of all points within the inner square.

3. NUMERICAL ANALYSIS. The generalization of conics (1) is closely related to the problem of finding the point of minimal sum of distances to three given points in the plane. According to Schreiber **[25]**, this problem was posed first by Pierre de Fermat at the end of his treatise on minima and maxima **[10]** and in a letter from Fermat to Torricelli. Soon several (geometrical) solutions were found to construct the point of minimal sum of distances, now called *Fermat-Torricelli point*, with ruler and compass. The generalizations that were introduced by different authors are straightforward: consider distance sums to more than three points, introduce weights for the focal points, and look at the problem in higher dimensions **[19]**. Whereas the Fermat-Torricelli point(s) minimize the distance sum, generalized conics might be viewed as point sets of constant distance sum or as equipotential lines of a surface $F := \{(\vec{x}, \rho(\vec{x})) | \vec{x} \in \mathbb{R}^d\} \subset \mathbb{R}^{d+1}$, the graph of the function ρ of d variables; see Figure 7. Related to this topic is the so-called *Steiner-Weber problem*: a given set of points is to be connected by a net of line segments of minimal total length; see **[18]** for a survey. For three (focal) points one has to determine their Fermat-Torricelli point and connect the latter with the focal points. Nevertheless, for more than three points the two problems differ **[25]**, **[26]**.

As with the ellipse, it is possible to draw all classical conics and generalized ellipses with a string construction if the weights are integers; see Figure 3. To

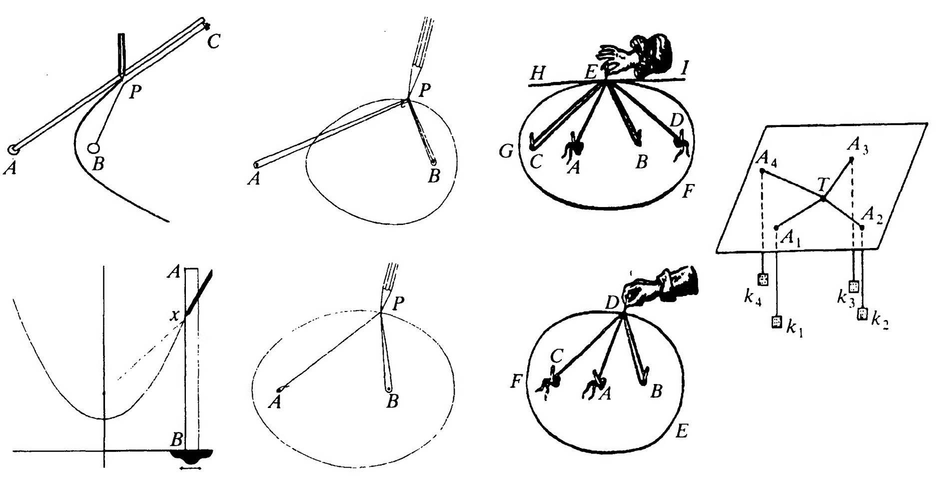

Figure 3. Constructions of classical and generalized conics with a string (from the left): parabola, hyperbola, ovals ([15], [16], [14]), generalized conics [29], and mechanical model to determine the Fermat-Torricelli point [11].

obtain the Fermat-Torricelli set and pictures of generalized conics for arbitrary configurations, it is possible to use computer programs and computer-algebra systems. First we describe how to get pictures from methods that are motivated by physics, and then we list commands for the computer-algebra systems Mathematica and Maple.

To determine a Fermat-Torricelli point for a configuration B, an intuitive idea is to use springs connecting $\vec{b}_i$ with a point $\vec{x} \in \operatorname{conv}\{B\}$, e.g., $\vec{x}^0 := \sum_{i=1}^n \alpha_i \vec{b}_i / \sum_{i=1}^n \alpha_i$ and try to minimize spring energy in this system. The spring force of an ideal spring is proportional to the extension of the spring from equilibrium, so this direct approach would be correct if we wanted to minimize the sum of *squares* of distances to the focal points. Here we need a constant force, so springs are not adequate. Nevertheless, in the 1-dimensional case we can use another mechanical model. We replace the focal points $\vec{b}_i \in \mathbb{R}^2$ by small holes through which we pass small ropes that carry a massive weight corresponding to the weight of the focus; see Figure 3 (right). This mechanical system constructs a Fermat-Torricelli point, because now all weights induce constant forces [**11**]. From this mechanical model we can derive a formula for iterative computation of Fermat-Torricelli point(s) in the 2-norm [**21**]:

$$\vec{x}^{i+1} = \frac{\sum_{j=1}^{n} \alpha_j \vec{b}_j \gamma_j^i}{\sum_{j=1}^{n} \alpha_j \gamma_j^i}, \quad \text{where } \gamma_j^i := \|\vec{x}^i - \vec{b}_j\|_2^{-1}. \tag{3}$$

Another way to compute a Fermat-Torricelli point numerically (provided, the norm is differentiable) is to descend along the gradient lines of the graph of F, as illustrated in Figure 4:

$$\vec{x}^{i+1} = \vec{x}^i - \vec{\nabla}(\rho(\vec{x}^i)). \tag{4}$$

Whereas the iteration (3) converges faster than (4) toward a Fermat-Torricelli point, the gradient method reveals geometric information about F and hence about its equipotential lines, the generalized conics, which are always perpendicular to the gradients. The gradient vanishes on the Fermat-Torricelli set. and the sequence of points $\vec{x}^i$ converges more slowly the closer it gets to the Fermat-Torricelli set. In [**22**] a Taylor expansion is used to show that there exists an approximating ellipse for generalized ellipses with $\rho \approx \underline{\rho}$. Since the gradient lines for (confocal) ellipses are confocal hyperbolas (Figure 4), close to the Fermat-Torricelli set we may use this approximating ellipse to complete the gradient lines of the generalized ellipses.

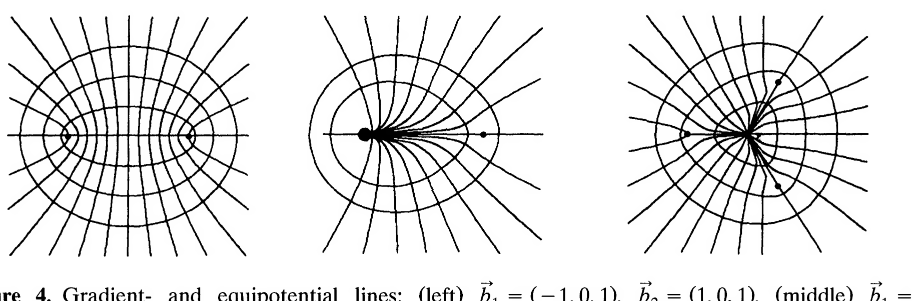

Figure 4. Gradient- and equipotential lines: (left) $\vec{b}_1 = (-1,0,1)$, $\vec{b}_2 = (1,0,1)$, (middle) $\vec{b}_1 = (-1,0,2)$, $\vec{b}_2 = (1,0,1)$, and (right) $\vec{b}_1 = (0,1,1)$, $\vec{b}_2 = (0.5, \sqrt{3}/2, 1)$, $\vec{b}_3 = (0.5, -\sqrt{3}/2, 1)$.

Drawing routines from computer-algebra systems such as Mathematica and Maple can be used to obtain pictures of generalized conics. With Mathematica it is possible to draw pictures for the 1-dimensional case, e.g., in the 2-norm, using the following commands; see Figures 6, 7:

```
(*Fig 6*)          r[x,y] = Sqrt[x^2 + (y + 1)^2] + Sqrt[x^2 + (y- a)^2]
(*Fig 7 left*)     r[x,y] = Sqrt[(x- 1)^2 + y^2] + Sqrt[(x + 1)^2 + y^2] + x
(*Fig 7 middle*)   r[x,y] = 2*Sqrt[x^2 + (y- 1)^2]- ... -Sqrt[(x + 1)^2 + y^2]
(*Fig 7 right*)    r[x,y] = Sqrt[x^2 + (y- 1)^2] + ... -Sqrt[(x + 1)^2 + y^2]
 Plot3D[r[x,y],{x,-2,2},{y,0,2}], ContourPlot[r[x,y],{x,-2,2},{y,0,2}]
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In addition, Maple can draw graphs of implicitly defined functions, so we can also get results in the 2-dimensional case; see Figure 5 (right).

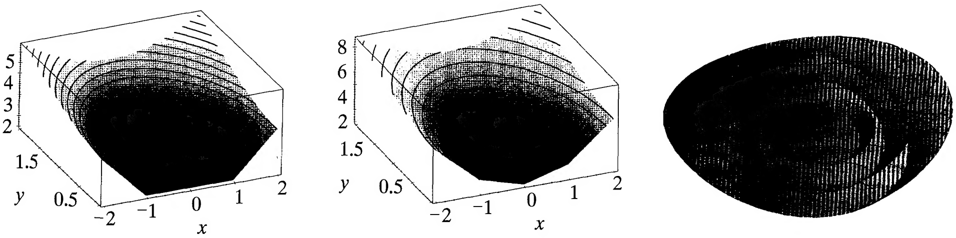

Figure 5. (Left) Niveau line plot for $\vec{b}_1 = (-1,0,1)$, $\vec{b}_2 = (1,0,1)$ (middle) $\vec{b}_1 = (-1,0,1)$, $\vec{b}_2 = (0,0,1)$, $\vec{b}_3 = (1,0,1)$ and (right) 2-dimensional generalized conics (ovaloides) for $\vec{b}_1 = (-1,0,0,1)$, $\vec{b}_2 = (1,0,0,2)$, $\rho = 3,4,5$.

4. ASYMPTOTIC BEHAVIOUR AND CLASSIFICATION. In general it is impossible to give an emplicit formula for the surface area or the volume of a generalized conic. Asymptotically, however, we can use expansions to approximate these values because for large ρ generalized ellipses converge to spheres, and in the 1-dimensional case for small ρ an approximating ellipse can be defined **[22]**.

From now on we use the Euclidean norm in $\mathbb{R}^2$, unless otherwise stated. This equips us with the corresponding scalar product. The classical conics can be obtained geometrically in several ways that reveal their intrinsic relationship. The parabola develops from an ellipse, if one fixes a focal point while moving the other along a line to infinity; see Figure 6. Since we cannot compute the distance to a point at infinity $\vec{b}_{i,\infty}$, we replace this point by a straight line g_i through the origin

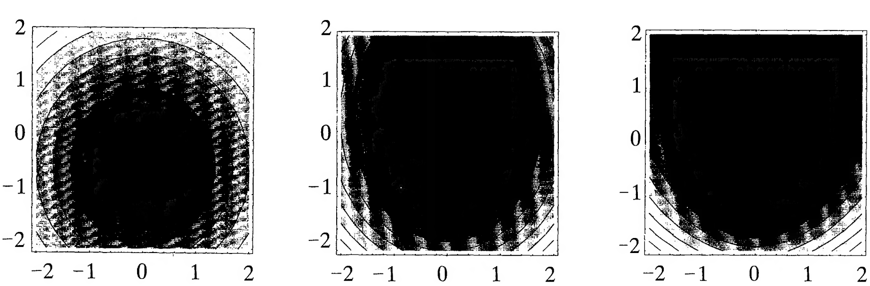

Figure 6. Moving a focus to infinity, $\vec{b}_1 = (0,-1,1)$, $\vec{b}_2 = (0,a,1)$ with $a = 1,5,10$.

whose unit normal vector $\vec{n}_i$ points to $\vec{b}_{i,\infty}$. Each point on g_i has the same 'distance' to $\vec{b}_{i,\infty}$, thus we replace the distance of a point $\vec{x}$ to $\vec{b}_{i,\infty}$ by the distance of $\vec{x}$ and the reference line g_i:

$$\rho\left(\vec{x}, \vec{b}_{i,\infty}\right) := -\langle \vec{n}_i, \vec{x} \rangle.$$

If we have several points at infinity, they can be combined to a single one. Let the configuration B contain m points at infinity with corresponding normal vectors $\vec{n}_i$ and weights α_i $(i = 1, \dots, m)$. Then the weighted sum of distances is

$$\sum_{i=1}^{m} \alpha_i \, \rho\left(\vec{x}, \vec{b}_{i,\infty}\right) = -\sum_{i=1}^{m} \alpha_i \langle \vec{n}_i, \vec{x} \rangle = -\left\langle \sum_{i=1}^{m} \alpha_i \vec{n}_i, \vec{x} \right\rangle = -\langle \alpha_0 \vec{n}_0, \vec{x} \rangle =: \rho\left(\vec{x}, \vec{b}_{0,\infty}\right),$$

where

$$\alpha_0 := \left\| \sum_{i=1}^{m} \alpha_i \vec{n}_i \right\|_2, \quad \text{and if } \alpha_0 \neq 0, \quad \vec{n}_0 := \frac{\sum_{i=1}^{m} \alpha_i \vec{n}_i}{\|\sum_{i=1}^{m} \alpha_i \vec{n}_i\|_2}.$$

For example, the points with a constant sum of weighted distances to two lines g_1, g_2 lie on parallels with normal vector $\alpha_1 \vec{n}_1 + \alpha_2 \vec{n}_2$.

Now we have $\rho(\vec{x}) = \rho_{\text{finite}}(\vec{x}) + \rho_\infty(\vec{x}) = \sum_{i=1}^{n} \alpha_i \|\vec{x} - \vec{b}_i\|_2 - \alpha_0 \langle \vec{n}_0, \vec{x} \rangle$. We use a Taylor expansion for ρ_{finite} for large $\|\vec{x}\|_2 \gg \|\vec{b}_i\|_2$ to obtain a classification of generalized conics in terms of generalized ellipses, parabolas, and hyperbolas, and a new case without classical counterpart. We use polar coordinates, $\vec{b}_i = \vec{b}_i(r_i, \vartheta_i)$, $\vec{x} = \vec{x}(r, \vartheta)$:

$$\rho_{\text{finite}}(\vec{x}) = \sum_{i=1}^{n} \alpha_i \|\vec{x} - \vec{b}_i\|_2 = \sum_{i=1}^{n} \alpha_i \sqrt{r^2 - 2 r r_i \cos(\vartheta - \vartheta_i) + r_i^2}\,.$$

The first order Taylor expansion for $r \gg r_i (\sqrt{1+x} \approx 1 + x/2 - x^2/8 + \cdots, x \ll 1)$ yields

$$\sum_{i=1}^{n} \alpha_i r \sqrt{1 - 2\frac{r_i}{r}\cos(\vartheta - \vartheta_i) + \frac{r_i^2}{r^2}} \approx r \sum_{i=1}^{n} \alpha_i =: r\alpha.$$

For $\vec{b}_{0,\infty}$ a point at infinity with weight α_0, we get from this expansion

$$\rho\left(\vec{x} \gg \vec{b}_i\right) \approx (\alpha - \alpha_0 \cos(\vartheta - \vartheta_0)) r. \tag{5}$$

If $\alpha \neq 0$ this is the polar representation of classical conics with a focal point as center. Depending on the ratio of weights of finite and infinity focal points, we thus get all types of classical conics:

$$r(\vartheta) = \frac{\rho_0}{\alpha - \alpha_0 \cos(\vartheta - \vartheta_0)} \begin{cases} \alpha_0 = 0 & \text{sphere}, \\ |\alpha| > |\alpha_0| & \text{ellipse or rotational ellipse}, \\ |\alpha| = |\alpha_0| & \text{parabola or rotational paraboloid}, \\ |\alpha| < |\alpha_0| & \text{hyperbola or rotational hyperboloid}. \end{cases} \tag{6}$$

Figure 7 shows generalized parabolas and hyperbolas and generalized conics that cannot be viewed as extensions of a classical type of conics. For $\alpha = 0$ and $\alpha_0 \neq 0$, (5) simplifies to $\rho(\vec{x} \gg \vec{b}_i) \approx -\alpha_0 \cos(\vartheta - \vartheta_0) r$ and the generalized conics converge to lines that are parallel to the (focal) line g_0.

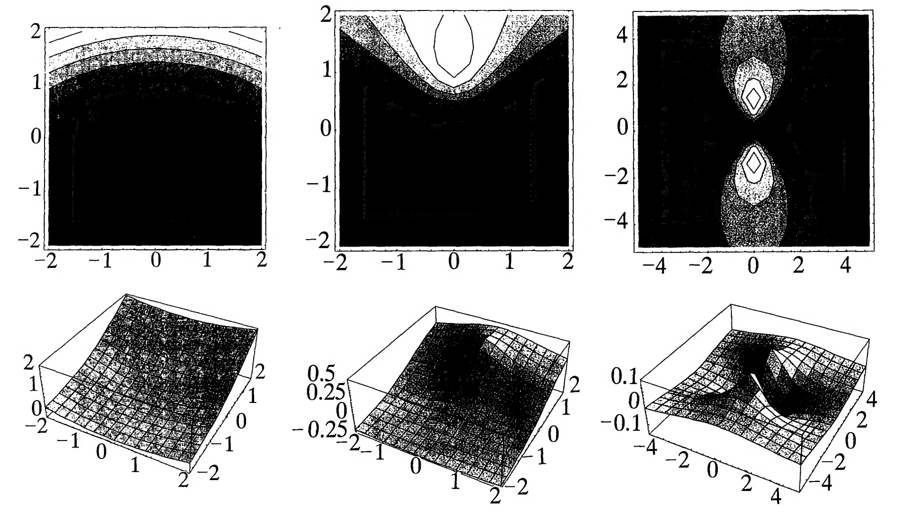

Figure 7. `ContourPlot` and `Plot3D` (from left to right): generalized parabolas, $\vec{b}_1 = (-1, 0, 1)$, $\vec{b}_2 = (1, 0, 1)$, $\vec{b}_3 = (0, -\infty, 2)$; generalized hyperbolas, $\vec{b}_1 = (-1, 0, -1)$, $\vec{b}_2 = (1, 0 - 1)$, $\vec{b}_3 = (0, 1, 2)$; non-classical generalized conics, $\vec{b}_1 = (-1, 0, -1)$, $\vec{b}_2 = (1, 0, -1)$, $\vec{b}_3 = (0, -1, 1)$, $\vec{b}_4 = (0, 1, 1)$.

If $\alpha = \alpha_0 = 0$ we do not get any information and have to expand to the second order:

$$\rho\left(\vec{x} \gg \vec{b}_i\right) \approx \left\langle \frac{\vec{x}}{r}, \vec{b} \right\rangle = \|\vec{b}\|_2 \cos \tilde{\vartheta},$$

where $\vec{b} := \sum_{i=1}^{n} \alpha_i \vec{b}_i$ and $\tilde{\vartheta}$ is the angle enclosed by $\vec{x}$ and $\vec{b}$. Hence for $\rho_0 > \|\vec{b}\|_2$, no unbounded solutions exist for $\rho(\vec{x}) = \rho_0$, i.e., the curves for $\rho_0 > \|\vec{b}\|_2$ are closed. Furthermore, when $\|\vec{b}\|_2 \neq 0$, the unbounded curves with $\rho_0 \leq \|\vec{b}\|_2$ tend to semi-lines defined by $\cos \tilde{\vartheta} = \rho_0 / \|\vec{b}\|_2$. Figure 7 shows an example ($\vec{b} = 0$) where nearly all curves are closed; only the curves for $\rho_0 = 0$ tend to straight lines $y = \pm x$ **[17]**.

In addition to the definitions of conics $K^{(1)}$ and $K^{(2)}$, 2-dimensional conics can be defined in several ways, e.g., ellipsoids as affine pictures of the unit sphere in 3-space ($a, b, c > 0$):

$$K_1^{(3)} := \left\{ \vec{x} \in \mathbb{R}^3 \,\middle|\, \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1 \right\}.$$

If a, b, c are all distinct, then this is not the geometrical locus of all points in 3-space that have a constant sum of Euclidean distances to two focal points! Nevertheless, if we define a norm $\|\cdot\|_\beta$ by

$$\|\vec{x}\|_\beta := \sqrt{(x_1/a)^2 + (x_2/b)^2 + (x_3/c)^2},$$

then $K_1^{(3)} = K_{1,1}$ with $\vec{b}_1 = \vec{0}$. Hence all ellipsoids are the geometrical locus of all points with a constant distance to one focal point, namely the origin, if we use an appropriate norm.

5. PHYSICAL PROPERTIES. Classical conics are interesting not only from a mathematical point of view but also for their *physical properties*. Any light ray sent out from one focus of an ellipse is reflected by the ellipse toward the other focus, a principle that is also known from architecture. This holds for all classical conics: for parabolas one focus is at infinity and for hyperbolas an imaginary focus exists; see Figure 8.

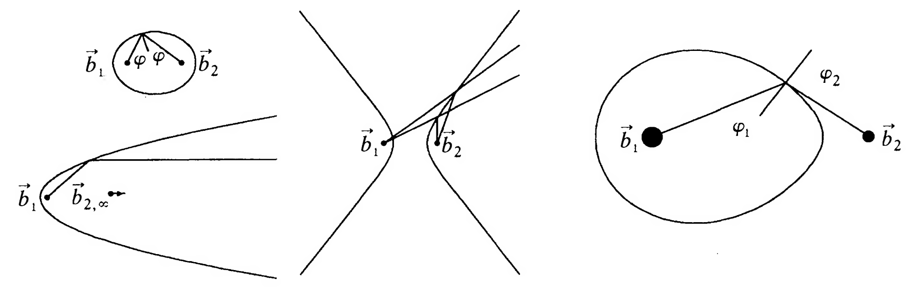

Figure 8. (Left) Reflexion of light rays emitted from a focus for classical conics $|\alpha_{1,2}| = 1$ and (right) Snell's law of refraction for an oval, $\vec{b}_1 = (-1, 0, 2)$, $\vec{b}_2 = (1, 0, 1)$.

When looking at generalized conics it is impossible to raise a similar question because we would have to divide an incident beam into several reflected ones if there are more than two foci. For $i = 1, \ldots, n$ let $\vec{l}_i$ be a light ray, emerging from a focus $\vec{b}_i$, that hits a generalized conic $K_{\rho,n}$ at $\vec{x}$ and makes an angle φ_i with the tangent $\vec{t}$ to $K_{\rho,n}$ at $\vec{x}$. The reflection property of ellipses ensures that $\varphi_1 = \pi - \varphi_2$, so $\cos\varphi_1 = -\cos\varphi_2$ and $\cos\varphi_1 + \cos\varphi_2 = 0$. An analog of this property, $\sum_{i=1}^{n} \alpha_i \cos\varphi_i = 0$, holds for generalized conics, and can be proved as follows (see Figure 9):

$$\cos\varphi_i = \frac{\vec{l}_i \vec{t}}{|\vec{l}_i|\,|\vec{t}|} = \frac{\left(\vec{x} - \vec{b}_i\right)\vec{t}}{|\vec{x} - \vec{b}_i|\,|\vec{t}|}$$

$$\Leftrightarrow \sum_{i=1}^{n} \alpha_i \frac{\left(\hat{x} - \vec{b}_i\right)\vec{t}}{|\vec{x} - \vec{b}_i|\,|\vec{t}|} = \frac{\vec{t}}{|\vec{t}|} \sum_{i=1}^{n} \alpha_i \frac{\left(\vec{x} - \vec{b}_i\right)}{|\vec{x} - \vec{b}_i|} = \frac{\vec{t}}{|\vec{t}|} \vec{\nabla}_x \rho(\vec{x}) \stackrel{!}{=} 0 \qquad (7)$$

which holds because niveau lines and gradients of the generalized radius are perpendicular at all points of $K_{\rho,n}$ where the gradient can be computed, i.e., at all points of $K_{\rho,n} \setminus B$.

This reflection property of generalized conics has no physical model except in the case of Cartesian ovals with two focal points [27]. One can think of the oval as a solid made out of a material with a refractive index n_1 (the surrounding material has index n_2). Then (7) is equivalent to Snell's law of refraction; see Figure 9, where we choose the indices of refraction to be $n_1 = \alpha_1$ and $n_2 = \alpha_2$.

Another reflection property of classical conics is the following: let $K_{\rho_1,2}$ and $K_{\rho_2,2}$ be two confocal ellipses, i.e., with the same focal points $\vec{b}_1, \vec{b}_2$ but different radii $\rho_1 < \rho_2$. Any light ray that is emitted along a tangent to the smaller ellipse is reflected from the inside of the bigger ellipse in such a way that the reflected beam is again a tangent to the smaller ellipse. This means that the two tangents to $K_{\rho_1,2}$ from any point $\vec{x} \in K_{\rho_2,2}$ make the same angle $\psi_1(\vec{x}) = \psi_2(\vec{x})$ with the normal to $K_{\rho_2,2}$ at $\vec{x}$. So far we have been unable to prove that this reflection property holds for generalized conics, but computer simulations indicate that it is valid only in

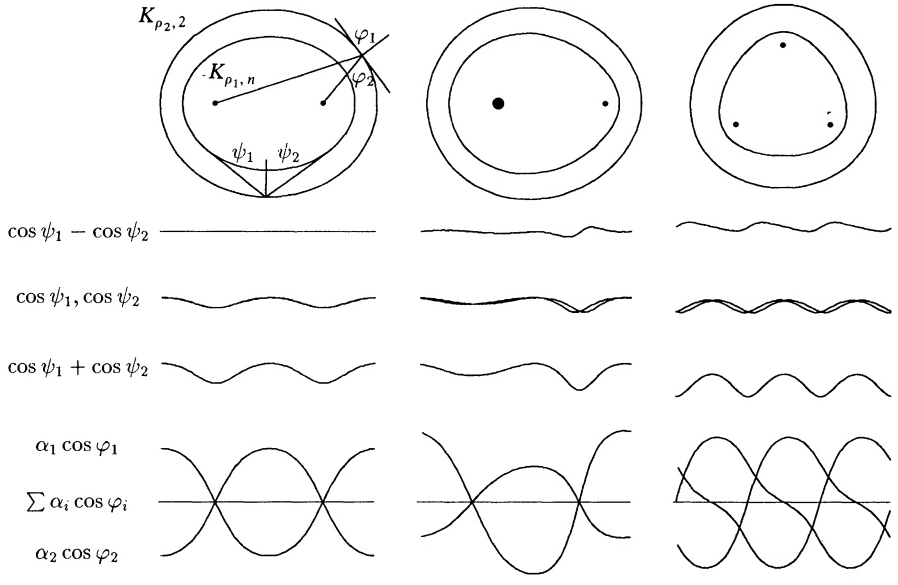

Figure 9. Angles from tangents as functions of the curve parameter (from the left): $\vec{b}_1 = (-1, 0, 1)$, $\vec{b}_2 = (1, 0, 1)$; $\vec{b}_1 = (-1, 0, 2)$, $\vec{b}_2 = (1, 0, 1)$; $\vec{b}_1 = (0, 1, 1)$, $\vec{b}_2 = (\sqrt{3}/2, -0.5, 1)$, $\vec{b}_3 = (-\sqrt{3}/2, -0.5, 1)$.

special cases. Figure 9 shows the angles $\psi_1(\vec{x})$, $\psi_2(\vec{x})$, and $\varphi_i(\vec{x})$ as functions of a curve parameter. The left figure in Figure 9 illustrates that $\psi_1 = \psi_2$ for an ellipse, whereas this identity is violated for an oval and a regular trillipse (a generalized ellipse where all focal points have equal weights and are located at the vertices of a regular n-gon).

Another interesting *mechanical* property of conics is that they provide solutions for the problem of constructing roads or counterwheels for conical wheels. For example, if we take an ellipse as a wheel that is fixed at its midpoint or at one of the focal points and we want to get a counterwheel that can roll in a fixed distance to the given ellipse, then for a certain distance k, the counterwheel is a congruent ellipse. This can be proved by solving the corresponding differential equations **[30]**. Moreover, we see that this property is also valid for hyperbolas because the solution does not depend on whether $\varepsilon < 1$ or $\varepsilon > 1$. One can show that from any pair of wheels given by polar representations $r_1(\varphi_1)$ and $r_2(\varphi_2)$, one can obtain a whole family of pairs of wheels by multiplying the angle function with a constant n. In order to get a periodic solution with $r_i(\varphi_i) = r_i(\varphi_i + 2\pi)$, we choose $n \in \mathbb{N}$:

$$r_{1/2,n}(\varphi_{1/2}) = \frac{k(1-\varepsilon^2)/2}{1 \pm \varepsilon\cos(n\varphi_{1/2})}, \quad \varphi \in [0, 2\pi). \tag{8}$$

This is another way to generalize conics; see Figure 10.

6. FINAL REMARKS. A natural question arising with different definitions for conics is whether they are equivalent. This question is related to a conjecture of Tschirnhaus **[29]** about representing all *geometrical curves* as *mechanical curves*:

$$\sum_{i+j\le n} a_{i,j}x^i y^j = 0 \overset{?}{\Leftrightarrow} \sum_{i=1}^{n} \alpha_i(a_{i,j})\left\|\vec{x} - \vec{b}_i(a_{i,j})\right\|_2 = \rho(a_{i,j}).$$

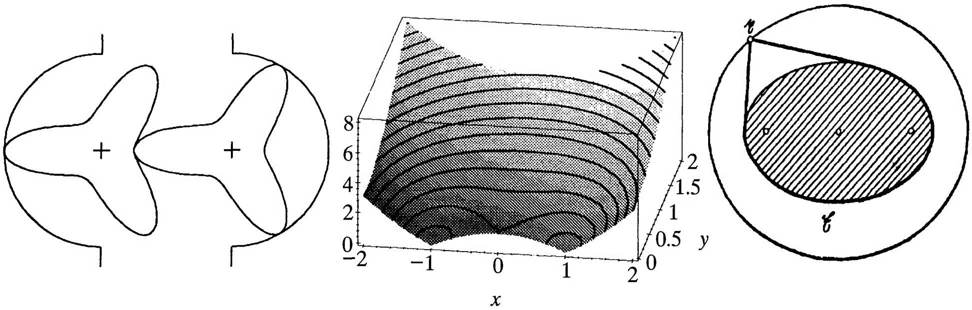

Figure 10. (Left) Pump made with congruent generalized ellipses in a case that is determined by the largest radius, (middle) Cassini ovals for $\vec{b}_1 = (-1, 0, 1)$, $\vec{b}_2 = (1, 0, 1)$, and (right) *rope property* of ellipses [3].

This conjecture is wrong in general; see **[9]** where, in addition, a construction method is given to find the locations of three focal points for the approximation of the cross section of a *real* egg.

The characterizations and classifications of generalized conics can be extended to other mathematical and physical properties and even other types of generalizations of conics. If we take not the sum of distances but their product, we get the so-called *Cassini ovals* (see Figure 10):

$$\sigma_n(\vec{x}) := \prod_{i=1}^{n} \|\vec{x} - \vec{b}_i\|^{\alpha_i} \Leftrightarrow \begin{array}{l} \tilde{K}_{\sigma,n} := \{\vec{x} \in \mathbb{R}^d | \sigma(\vec{x}) = \sigma\}, \\ \tilde{E}_{\sigma,n} := \{\vec{x} \in \mathbb{R}^d | \sigma(\vec{x}) \leq \sigma\}. \end{array} \tag{9}$$

We close with a few open questions on generalized conics:

- If we take two focal points $\vec{b}_1, \vec{b}_2$ with weights $\alpha_1 = 1$, $\alpha_2 = \pm 1$, then we get confocal ellipses and hyperbolas, and the hyperbolas are perpendicular to the ellipses; see Figure 4. Do there exist confocal generalized conics with such a property?
- Using a rope of length larger than the circumference of a given ellipse, it is possible to draw a confocal ellipse; see Figure 10 (right). Is this also possible for generalized ellipses?
- Which results can be extended to the cases of infinitely many focal points or to continuous sets of foci ($\alpha(\vec{x})$ a weight function):

$$E_{\rho,G} := \left\{ \vec{x} \in \mathbb{R}^d \middle| \int_G \alpha(\vec{g}) \|\vec{x} - \vec{g}\| \, dg \leq \rho \right\}, \quad G \subset \mathbb{R}^d?$$

- Regular generalized ellipses have a shape that is very similar to that of curves of constant breadth or so-called *orbiforms*. Are there generalized ellipses (in d-space) that have the orbiform property?

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